Reducing Graph Coloring to SAT

A *k-coloring* of a graph is a labelling of its vertices with at most k colors such that no two vertices sharing the same edge have the same color. The problem of generating a k-coloring of a graph (V, E) can be reduced to SAT as follows. For every $v \in V$ and every $i \in \{1, \dots, k\}$, introduce an atom p_{vi}. Intuitively, this atom expresses that vertex v is assigned color i. Consider the following propositional formulas:

$$\begin{array}{ll} \bigvee_{1 \le i \le k} p_{vi} & (v \in V), \\ \neg(p_{vi} \wedge p_{vj}) & (v \in V,\ 1 \le i < j \le k), \\ \neg(p_{vi} \wedge p_{wi}) & (\{v, w\} \in E,\ 1 \le i \le k). \end{array} \tag{1}$$

The interpretations satisfying these formulas are in a 1–1 correspondence with k-colorings of (V, E).

Problem 3. (a) Write out formulas (1) for the graph

$$A \text{———} B \text{———} C$$

and $k = 2$. (Suggestion: use the abbreviation p_{A1} for $A1$, and similarly for the other atoms.) (b) We would like to find a k-coloring of a graph (V, E) such that color 1 is assigned to at most one vertex. Modify formulas (1) accordingly.

Problem 4. Use DPLL to find (a) a 2-coloring of the graph from Problem 3; (b) a 2-coloring of that graph such that color 1 is assigned to at most one vertex.