

## Introduction to Iterative Solutions for Nonlinear System

The systems of equations that we have been studying so far have been *linear*, i.e. we can write them in the matrix form  $\mathbf{Ax} = \mathbf{b}$ . However, many applications which lead to scientific computing involve nonlinear systems of equations. An example of a simple nonlinear system is

$$x_1^2 + x_2^2 - x_1 = 0 \quad (1)$$

$$x_1^2 - x_2^2 - x_2 = 0 \quad (2)$$

More generally, a nonlinear system for  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)^T$  has the form

$$f_1(x_1, x_2, \dots, x_n) = 0$$

$$f_2(x_1, x_2, \dots, x_n) = 0$$

$$\vdots$$

$$f_n(x_1, x_2, \dots, x_n) = 0.$$

If we define the vector-valued function  $\mathbf{F}$  as

$$\mathbf{F}(x_1, x_2, x_3, \dots, x_n) = (f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n))$$

the most general form in which we denote nonlinear systems is

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

Finding the solution to nonlinear systems can be more complicated than finding solutions to linear systems. One reason for this is illustrated in Figure 1. As one can see from this figure, the graphs of Equations (1) and (2) cross at two coordinate points  $(x_1, x_2)$ , which means that the system has two solutions. This illustrates one issue that often comes up when numerically solving nonlinear systems: How can you tell if you are converging to the “correct” solution? We will be discussing this further in class.

There are many computational methods that can be used to solve nonlinear systems, two of which we already discussed for linear systems. These are the Jacobi Method and the Gauss-Seidel method, which both rely on fixed-point iteration (in fact, for nonlinear systems the Jacobi Method is called fixed-point iteration). By definition, a function  $\mathbf{G} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  has a *fixed point* at  $\mathbf{p} \in D$  if

$$\mathbf{G}(\mathbf{p}) = \mathbf{p}.$$

To take advantage of the concept of a fixed point one must reformulate the original system  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$  into  $\mathbf{G}(\mathbf{x}) = \mathbf{x}$ . Looking back at our simple example system defined by Equations (1) and (2), one could rewrite the system as

$$\begin{aligned} x_1 &= x_1^2 + x_2^2 \\ x_2 &= x_1^2 - x_2^2. \end{aligned}$$

In this case, the function  $\mathbf{G}$  is defined as

$$\mathbf{G}(\mathbf{x}) = (g_1(x_1, x_2), g_2(x_1, x_2))^T = (x_1^2 + x_2^2, x_1^2 - x_2^2)^T. \quad (3)$$

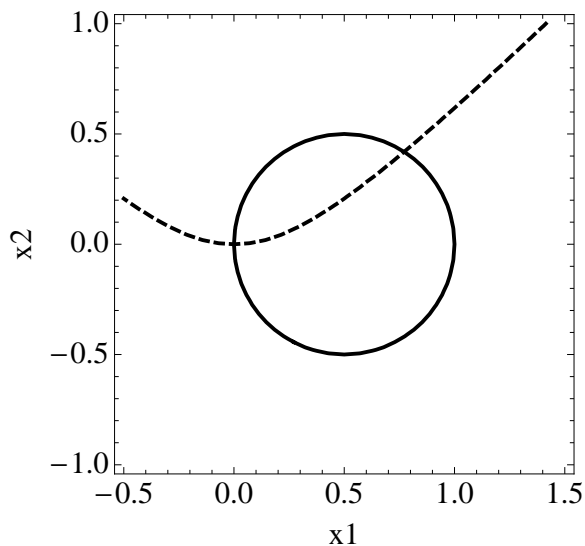


Figure 1: Graph of system represented by Equations (1) and (2).

Alternatively, one could rewrite the system as

$$\begin{aligned}x_1 &= \sqrt{x_1 - x_2^2} \\x_2 &= \sqrt{x_1^2 - x_2},\end{aligned}$$

in which case

$$\mathbf{G}(\mathbf{x}) = (\sqrt{x_1 - x_2^2}, \sqrt{x_1^2 - x_2}). \quad (4)$$

It turns for reasons that will be discussed in class that Equation 4 provides a better form for  $\mathbf{G}$ .

Once an appropriate form for  $\mathbf{G}$  is determined, fixed-point iteration and the Gauss-Seidel method for nonlinear problems are very similar to the methods for linear problems. For fixed-point iteration, given an initial guess  $\mathbf{x}^{(0)}$ , you iterate according to

$$\mathbf{x}^{(k+1)} = \mathbf{G}(\mathbf{x}^{(k)}) \quad (5)$$

until  $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty$  or  $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_2$  is within a predetermined tolerance. For Gauss-Seidel, you use the updated value of the previously computed  $(x_1^{(k+1)}, x_2^{(k+1)}, \dots, x_i^{(k+1)})^T$  in the update of  $x_{i+1}$  such that

$$x_{i+1}^{(k+1)} = g_{i+1}(x_1^{(k+1)}, x_2^{(k+1)}, \dots, x_i^{(k+1)}, x_{i+1}^{(k)}, \dots, x_n^{(k)}).$$

Note that these two procedures are almost identical to what you did for linear systems. The primary difference in the algorithm is that your function  $\mathbf{G}$  is nonlinear. Another important point to consider is that your choice of initial guess, i.e.  $\mathbf{x}^{(0)}$ , is extremely important in determining which solution you converge to, or whether you converge to a solution at all.

**Assignment:**

1. Carefully read sections 10.1 and 10.2 in *Numerical Analysis* by Burden and Faires.
2. Consider the nonlinear system

$$\begin{aligned}x_1^2 - 10x_1 + x_2^2 + 6 &= 0 \\x_1x_2^2 + x_1 - 10x_2 + 8 &= 0.\end{aligned}$$

The corresponding fixed point problem is

$$\begin{aligned}x_1 &= g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 6}{10} \\x_2 &= g_2(x_1, x_2) = \frac{x_1x_2^2 + x_1 + 8}{10}\end{aligned}$$

- (a) Plot the original nonlinear system (you can do this in Mathematica). How many solutions does this system have?
- (b) Use the graph from part (a) as an initial approximation for fixed-point iteration. Please do this in Matlab. Make sure that you include appropriate stopping criteria (i) for convergence and (ii) if you exceed a maximum number of iterations. How many iterations do you need so that the difference between two iterates in the  $L_\infty$  norm is within  $10^{-5}$ ?
- (c) What happens when you change your initial guess in an attempt to converge to another solution of the system?
- (d) Use Gauss-Seidel to approximate a solution to the system (again in Matlab). How many iterations do you need so that the difference between two iterates in the  $L_\infty$  norm is within  $10^{-5}$ ?