conditions

$$P(2,x_{n+1}) = Q_f \qquad \frac{\partial P}{\partial x_{n+1}}(2,x_{n+1}) = 0$$
$$S(2,x_{n+1}) = M_f \qquad \frac{\partial S}{\partial x_{n+1}}(2,x_{n+1}) = 0 \qquad (31)$$
$$T(2,x_{n+1}) = W_f \qquad \frac{\partial S}{\partial x_{n+1}}(2,x_{n+1}) = 0$$

and substituting in equation (24) and using algorithm 1, the desired optimal control problem can be solved [1].

Remark: By a little change, the method presented above can be generalized to the case in which the number of switching is more than one.

III. Modeling and Optimal Control of Two-Tank System

Two-tank system has been studied by many researchers as an appropriate system for investigating hybrid systems, since one can evaluate efficiency of different methods by increasing the number of tanks [14]-[16]. For example, modeling and optimization of the system based on hybrid automata and an innovative optimization approach are proposed in [14]. In this paper we apply the mentioned optimal method to a two-tank system by modeling and appropriately linearizing it.

A two-tank system as shown in figure 1 is composed of two tanks connected to each other. The tanks are filled with fluids and the fluids are controlled by three control valves. At the beginning, tanks 1 and 2 are disconnected and at the switching instance (which should be determined by us) they will be connected and the fluid flows from tank 1 to tank 2. The goal is to take fluid level in the tanks at a predetermined value. Achieving this goal together with satisfying the existing constraints necessitate appropriate selection of the cost function.

The rate of the level of fluid in each tank is related directly to the difference between inflow and outflow rates and inversely to tank cross section area. Thus, nonlinear dynamics of the systems can be expressed by following equations:

$$\dot{x}_1 = \frac{1}{A_1}(F_1 - F_2) \qquad (32)$$

$$\dot{x}_2 = \frac{1}{A_2}(F_2 - F_3) \qquad (33)$$

$$F_1 = k_1 u_1 \qquad (34)$$

$$F_2 = \begin{cases} 0 & t < t_1 \\ k_2 u_2 \sqrt{x_1} & t \geq t_1 \end{cases} \qquad (35)$$

$$F_3 = k_3 u_3 \sqrt{x_2} \qquad (36)$$

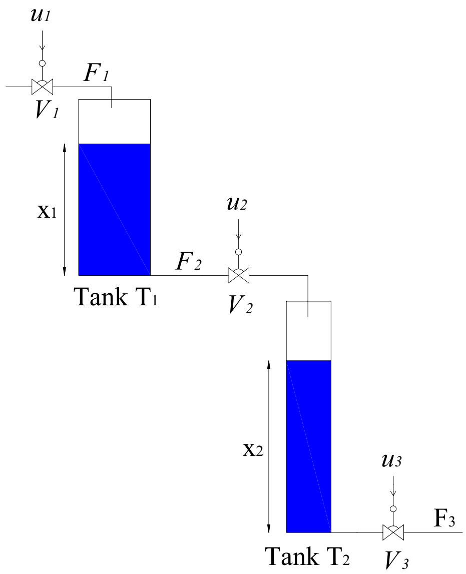

Fig. 1 Two-tank system

where, x_1 and x_2 the are height of fluid in each tank, u_i is the control signal for valve V_i, A_i the cross section area of i'th tank and k_i is valve constant for i'th valve. It is assumed that control signals and state variables can take values in the intervals $[u_{\min}, u_{\max}]$ and $[x_{\min}, x_{\max}]$ respectively. The mentioned constraints and goal can be taken into account aptly in the cost function so that obtained solution meets the desired conditions. For example, if the goal is taking state variables x_1 and x_2 (fluid heights of the tanks 1 and 2) to values x_{1n} and x_{2n}, then $x_1(t_f)$ and $x_2(t_f)$ can get the desired values by adding

$$\alpha[(x_1(t_f) - x_{1n})^2 + (x_2(t_f) - x_{2n})^2]$$

to the cost function and setting α a great value.

Nonlinear dynamics of the system can be expressed explicitly in two distinct regions:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \frac{1}{A_1}(k_1 u_1) \\ -\frac{1}{A_2}(k_3 u_3 \sqrt{x_2}) = 0 \end{cases} \qquad t < t_1 \qquad (37)$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \frac{1}{A_1}(k_1 u_1 - k_2 u_2 \sqrt{x_1}) \\ \frac{1}{A_2}(k_2 u_2 \sqrt{x_1} - k_3 u_3 \sqrt{x_2}) \end{cases} \qquad t \geq t_1 \qquad (38)$$

By linearizing the above equations around the equilibrium point (x_{eq}, u_{eq}) (in both region) using

$$\dot{x} = \left(\frac{\partial f}{\partial x}\right)\Bigg|_{\substack{x=x_{eq}\\u=u_{eq}}} (x - x_{eq}) + \left(\frac{\partial f}{\partial u}\right)\Bigg|_{\substack{x=x_{eq}\\u=u_{eq}}} (u - u_{eq}) \tag{39}$$

and defining new variables $x^* = x - x_{eq}, u^* = u - u_{eq}$ and expressing the cost function as a function of x^* and u^*, the mentioned problem is converted to a linear switching problem with quadratic cost function. The latter problem can be solved using the mentioned method such that the cost function is minimized and thereby main goal of control together with the problem's constraints are satisfied.

IV. Simulation Results

The desired heights of fluid in the tanks are $x_{1n} = 0.5m$ and $x_{2n} = 0.1m$ respectively in simulations and therefore we add term $140000[(x_1(t_f) - 0.5)^2 + (x_2(t_f) - 0.1)^2]$ to the cost function. In order to achieve the goal of control and satisfying problem constraints, matrices in the cost function are chosen to be:

$$Q_f = 14 \times 10^4 \times I_{2\times2} \qquad M_f = 10^4 \times [-3.5 \quad -1.26]$$
$$W_f = 4942 \qquad Q = 0_{2\times2} \qquad V = 0_{2\times3} \qquad R = 50 \times I_{3\times3}$$
$$M = [-25 \quad -10] \quad N = [-5 \quad -5 \quad -15] \quad W = 19.4$$

Simulation results are shown in figures 2, 3 and 4. According to the above matrices and by choosing $x_{2n} = 0.1$, $x_{1n} = 0.5$, $t_f = 10$, $t_0 = 0$, the optimal switching instance, $x_1(t_f)$ and $x_2(t_f)$ would be 2.9176s, 0.5019m and 0.1076m respectively. It can be seen that the two last values are very close desired values, i.e. 0.5 and 0.1. Figures 2 and 3 show that until instance t=2.9176s, height of the first tank increases to about 1.62 m while height of the second tank is zeros.

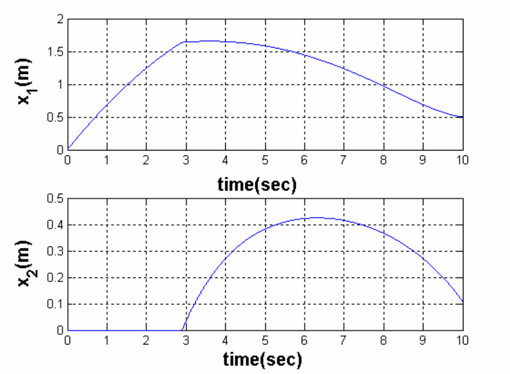

Fig. 2 Height of fluid in the first and second tanks

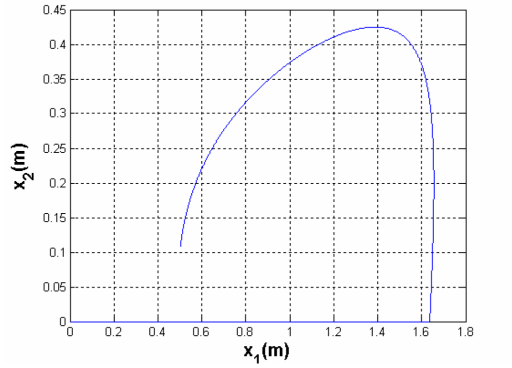

Fig. 3 State trajectory

After the switching instance till t=6.2081s, height of the first tank decreases and fluid inside the second tank is raised and after that heights of both the tanks decrease. Control inputs are shown in figure 4.

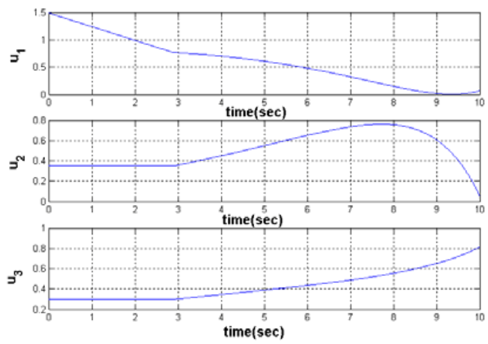

Fig. 4 Control inputs

V. Conclusion

In this paper, we considered the optimal control of linear switching systems with quadratic cost function which are a class of hybrid systems. The explained method was applied on the two-tank system which is an appropriate system for modeling and control of hybrid systems. In this method, the problem was converted to a conventional optimal control problem using the parameterization of switching instances and the switching instance and optimal input were obtained by algorithm 1. Moreover, in this method, the goal of control and also existing constraints are satisfied by appropriate selection of the matrices in cost function. Simulation results demonstrate that the explained method is appropriate for optimal control of linear switching systems.

Acknowledgment

We would like to thank Mr. J. Habibi from CIPCE, University of Tehran and also P. Sakian, M Ghandi and M. J. Abdoli from Multimedia Lab, SUT for their willing and consistent help through preparing the paper.

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