Hybrid Modeling and Optimal Control of a Two-Tank System as a Switched System

H. Mahboubi, B. Moshiri, and A. Khaki Seddigh

Abstract**—**In the past decade, because of wide applications of hybrid systems, many researchers have considered modeling and control of these systems. Since switching systems constitute an important class of hybrid systems, in this paper a method for optimal control of linear switching systems is described. The method is also applied on the two-tank system which is a much appropriate system to analyze different modeling and control techniques of hybrid systems. Simulation results show that, in this method, the goals of control and also problem constraints can be satisfied by an appropriate selection of cost function.

Keywords**—**Hybrid systems, optimal control, switched systems, two-tank system

I. Introduction

In the context of modeling and control, combinational systems are systems which are constituted from a combination of continuous and discrete elements and their behavior is a result of mutual effects of these elements on each other. In past, dynamics of such systems was considered separately.

When continuous and discrete elements are working together in a process and there is a considerable relation between these elements, it is needed to consider dynamical elements and their mutual relations altogether to get to a thorough understanding of the system's behavior and achieve high efficiency. This is the only way to exactly analyze and optimize a process. That is why in the last years many researchers have concentrated their efforts on modeling and control of hybrid systems. However, general methods for analysis and design of hybrid systems have not been developed yet.

It is noteworthy that switched systems are an important part of hybrid systems and consist of some subsystems and a switching law which specifies the active subsystem in each time instance. Many industrial systems such as chemical systems, transportation systems, etc. can be modeled as a switched system [1].

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For optimal control of switched systems it is necessary to obtain the optimal input and optimal switching instances simultaneously. In this paper, the two-tank system is considered as a switched system and using quadratic cost function, the optimal switching instance and optimal input are obtained such that the cost function is minimized

II. Optimal Control of Switched Systems

In this paper, it is assumed that switched system consists of the subsystems

$$\dot{x} = f_i(x,u),\, f_i : R^n \times R^m \to R^n,\, i \in I = \{1,2,...,M\} \quad (1)$$

In order to control switched systems it is necessary to obtain switching sequences in addition to the input [2]-[7]. In fact, the switching sequence represents the sequence of active subsystems and is defined as

$$\sigma = \big((t_0,i_0),(t_1,i_1),...,(t_K,i_K)\big) \quad (2)$$

where $i_k \in I \ (k = 0,1,...,K)$, $0 \le K < \infty$ and $t_0 \le t_1 \le ... \le t_K \le t_f$. The pair (t_k, i_k) shows that we switch in t_k from subsystem i_{k-1} to subsystem i_k. As mentioned before, for optimal control of the switched system one must obtain optimal input and optimal switching time simultaneously. The General Switched Linear Quadratic systems constitute an important class of switched systems whose optimal control method is described as follows:

Problem 1:

Suppose the following switched system

$$\begin{aligned} \dot{x} &= A_1 x + B_1 u \qquad t_0 \le t < t_1 \\ \dot{x} &= A_2 x + B_2 u \qquad t_1 \le t \le t_f \end{aligned} \quad (3)$$

The main goal is determination of switching time t_1 and input $u(t)$ such that the following cost function is minimized:

$$\begin{aligned} J_1 &= \frac{1}{2} x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f + \\ &\int_0^{t_f} \Big(\frac{1}{2} x^T Q x + x^T V u + \frac{1}{2} u^T R u + Mx + Nu + W\Big) dt \end{aligned} \quad (4)$$

Where $Q_f, M_f, W_f, Q, V, R, M, N, W$ are matrices with appropriate dimensions and $R > 0, Q \ge 0, Q_f \ge 0$ [8].

In order to solve the above problem, it is divided to two stages. In the first stage, a sequence of switching instances is considered and the minimum cost function with respect to input u is obtained. In the second stage, using the values obtained in the first stage, switching instances are modified such that the cost function approaches its minimum value [9]-[11].The following numerical algorithm is used for implementing this optimization method:

ALGORITHM 1:

1. Set the iteration index $j = 0$ and initialize switching instances $\hat{t}_j$.

2. Calculate $J_1(\hat{t}_j)$ by solving the optimal control problem (according to stage 1).

3. Calculate $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}_j)$.

4. Change $\hat{t}_j$ to $\hat{t}_{j+1} = \hat{t}_j + \alpha^j d\hat{t}_j$ using the value calculated in previous iteration (α^j should be chosen such that desired convergence is attained)

5. Repeat steps 2, 3 and 4 until the norm of projection of $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}_j)$ is smaller than a given small value.

According to the above algorithm, the values of $J_1(\hat{t}_j)$ and $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}_j)$ are needed. To calculate these values and also convert Problem 1 to a conventional optimal control problem, method of parameterization of the switching instances is deployed as follows:

A new state variable x_{n+1} is defined:

$$x_{n+1} = t_1 \quad , \quad \frac{dx_{n+1}}{dt} = 0 \tag{5}$$

A new independent variable τ is also defined as follows

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau & 0 \le \tau < 1 \\ x_{n+1} + (t_f - x_{n+1})(\tau - 1) & 1 \le \tau \le 2 \end{cases} \tag{6}$$

It is clear that according to the above definition, $\tau = 0$, $\tau = 1$ and $\tau = 2$ correspond to $t = t_0$, $t = t_1$ and $t = t_f$, respectively[12]. Considering τ as time variable and defining x_{n+1} Problem 1 is converted to Problem 2:

Problem 2:

For system with dynamics

$$\begin{aligned} &\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)(A_1 x + B_1 u) \qquad 0 \le \tau < 1 \\ &\frac{dx_{n+1}}{d\tau} = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{dx(\tau)}{d\tau} = (t_f - x_{n+1})(A_2 x + B_2 u) \qquad 1 \le \tau \le 2 \\ &\frac{dx_{n+1}}{d\tau} = 0 \end{aligned} \tag{8}$$

It is desired to calculate $u(\tau)$ and x_{n+1} in the interval $\tau \in [0,2]$ such that the following cost function is minimized:

$$\begin{aligned} J_1 = &\frac{1}{2} x(2)^T Q_f\, x(2) + M_f x(2) + W_f + \\ &\int_0^1 (x_{n+1} - t_0) L(x,u) d\tau + \int_1^2 (t_f - x_{n+1}) L(x,u) d\tau \end{aligned} \tag{9}$$

where

$$L(x,u) = \frac{1}{2} x^T Q x + x^T V u + \frac{1}{2} u^T R u + Mx + Nu + W \tag{10}$$

Conventional methods can be used for solving this problem. Assume that the optimal value function is

$$V^* = \frac{1}{2} x^T P(\tau, x_{n+1}) x + S(\tau, x_{n+1}) x + T(\tau, x_{n+1}) \tag{11}$$

where $P^T(\tau, x_{n+1}) = P(\tau, x_{n+1})$. The HJB equation is

$$\begin{aligned} -\frac{\partial V^*}{\partial \tau}(x, \tau, x_{n+1}) = \min_{u(\tau)} \{ &(x_{n+1} - t_0)(L(x,u) \\ &+ \frac{\partial V^*}{\partial x}(x, \tau, x_{n+1}) f_1(x,u))\} \end{aligned} \tag{12}$$

in the interval $\tau \in [0,1)$ and

$$\begin{aligned} -\frac{\partial V^*}{\partial \tau}(x, \tau, x_{n+1}) = \min_{u(\tau)} \{ &(t_f - x_{n+1})(L(x,u) \\ &+ \frac{\partial V^*}{\partial x}(x, \tau, x_{n+1}) f_2(x,u))\} \end{aligned} \tag{13}$$

in the interval $\tau \in [1,2]$ [1].

Using a method similar to solving LQR problem [13], the solution for HJB equation is as follows:

$$u(x, \tau, x_{n+1}) = -K(\tau, x_{n+1}) x(\tau, x_{n+1}) - E(\tau, x_{n+1}) \tag{14}$$

$$K(\tau, x_{n+1}) = R^{-1}(B_k^T P(\tau, x_{n+1}) + V^T) \tag{15}$$

$$E(\tau, x_{n+1}) = R^{-1}(B_k^T S^T(\tau, x_{n+1}) + N^T) \tag{16}$$

In the above equation, indices $k = 1$ and $k = 2$ correspond to the intervals $\tau \in [0,1)$ and $\tau \in [1,2]$, respectively. $P(\tau, x_{n+1})$, $S(\tau, x_{n+1})$ and $T(\tau, x_{n+1})$ which are denoted respectively by P, S and T satisfy the following Riccati equation:

$$\begin{aligned} -\frac{\partial P}{\partial \tau} = (x_{n+1} - t_0)(&Q + PA_1 + A_1^T P \\ &- (PB_1 + V) R^{-1} (B_1^T P + V^T)) \end{aligned} \tag{17}$$

$$\begin{aligned} -\frac{\partial S}{\partial \tau} = (x_{n+1} - t_0)(&M + SA_1 \\ &- (N + SB_1) R^{-1} (B_1^T P + V^T)) \end{aligned} \tag{18}$$

$$\begin{aligned} -\frac{\partial T}{\partial \tau} = (x_{n+1} - t_0)(&W - \frac{1}{2}(N + SB_1) \\ &\times R^{-1} (B_1^T S^T + N^T)) \end{aligned} \tag{19}$$

in the interval $\tau \in [0,1)$ and

$$-\frac{\partial P}{\partial \tau}=(t_f-x_{n+1})(Q+PA_2+A_2^TP-(PB_2+V)R^{-1}(B_2^TP+V^T)) \tag{20}$$

$$-\frac{\partial S}{\partial \tau}=(t_f-x_{n+1})(M+SA_2-(N+SB_2)R^{-1}(B_2^TP+V^T)) \tag{21}$$

$$-\frac{\partial T}{\partial \tau}=(t_f-x_{n+1})(W-\frac{1}{2}(N+SB_2)\times R^{-1}(B_2^TS^T+N^T)) \tag{22}$$

in the $\tau\in[1,2]$.

Along with the boundary equations $P(2,x_{n+1})=Q_f$, $S(2,x_{n+1})=M_f$ and $T(2,x_{n+1})=W_f$, (17-22) can be solved (for a fixed x_{n+1}) backward in τ and obtain the parameterized optimal cost at $\tau=0$.

$$\begin{aligned}J_1(t_1)&=J_1(x_{n+1})=V^*(x_0,0,x_{n+1})\\&=\frac{1}{2}x_0^TP(0,x_{n+1})x_0+S(0,x_{n+1})x_0+T(0,x_{n+1})\end{aligned} \tag{23}$$

and from the above equation we have

$$\begin{aligned}\frac{dJ_1}{dx_{n+1}}(x_{n+1})&=\frac{\partial V^*}{\partial x_{n+1}}(x_0,0,x_{n+1})\\&=\frac{1}{2}x_0^T\frac{\partial P}{\partial x_{n+1}}(0,x_{n+1})x_0+\frac{\partial S}{\partial x_{n+1}}(0,x_{n+1})x_0\\&+\frac{\partial T}{\partial x_{n+1}}(0,x_{n+1})\end{aligned} \tag{24}$$

for obtaining $\frac{dJ_1}{dx_{n+1}}$ using the above equation, values of $\frac{\partial T}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$, $\frac{\partial P}{\partial x_{n+1}}$ should be obtained in $(0,x_{n+1})$. Differentiating equations (17-22) with respect to x_{n+1} the mentioned values are obtained as follows:

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial P}{\partial x_{n+1}}&=(Q+PA_1+A_1^TP-(PB_1+V)R^{-1}(B_1^TP+V^T))\\&+(x_{n+1}-t_0)(\frac{\partial P}{\partial x_{n+1}}A_1+A_1^T\frac{\partial P}{\partial x_{n+1}}-(\frac{\partial P}{\partial x_{n+1}}B_1)\\&\times R^{-1}(B_1^TP+V^T)-(PB_1+V)R^{-1}(B_1^T\frac{\partial P}{\partial x_{n+1}}))\end{aligned} \tag{25}$$

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial S}{\partial x_{n+1}}&=(M+SA_1-(N+SB_1)R^{-1}(B_1^TP+V^T))\\&+(x_{n+1}-t_0)(\frac{\partial S}{\partial x_{n+1}}A_1-(\frac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^TP+V^T)\\&-(N+SB_1)R^{-1}(B_1^T\frac{\partial P}{\partial x_{n+1}}))\end{aligned} \tag{26}$$

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial T}{\partial x_{n+1}}&=(W-\frac{1}{2}(N+SB_1)R^{-1}(B_1^TS^T+N^T))\\&+(x_{n+1}-t_0)(-\frac{1}{2}\frac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^TS^T+N^T)\\&-\frac{1}{2}(N+SB_1)R^{-1}B_1^T(\frac{\partial S}{\partial x_{n+1}})^T)\end{aligned} \tag{27}$$

in the interval $\tau\in[0,1)$ and

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial P}{\partial x_{n+1}}&=-(Q+PA_2+A_2^TP-(PB_2+V)R^{-1}(B_2^TP+V^T))\\&+(t_f-x_{n+1})(\frac{\partial P}{\partial x_{n+1}}A_2+A_2^T\frac{\partial P}{\partial x_{n+1}}-(\frac{\partial P}{\partial x_{n+1}}B_2)\\&\times R^{-1}(B_2^TP+V^T)-(PB_2+V)R^{-1}(B_2^T\frac{\partial P}{\partial x_{n+1}}))\end{aligned} \tag{28}$$

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial S}{\partial x_{n+1}}&=-(M+SA_2-(N+SB_2)R^{-1}(B_2^TP+V^T))\\&+(t_f-x_{n+1})(\frac{\partial S}{\partial x_{n+1}}A_2-(\frac{\partial S}{\partial x_{n+1}}B_2)R^{-1}(B_2^TP+V^T)\\&-(N+SB_2)R^{-1}(B_2^T\frac{\partial P}{\partial x_{n+1}}))\end{aligned} \tag{29}$$

$$\begin{aligned}-\frac{\partial}{\partial \tau}\frac{\partial T}{\partial x_{n+1}}&=(W-\frac{1}{2}(N+SB_2)R^{-1}(B_2^TS^T+N^T))\\&+(t_f-x_{n+1})(-\frac{1}{2}\frac{\partial S}{\partial x_{n+1}}B_2)R^{-1}(B_2^TS^T+N^T)\\&-\frac{1}{2}(N+SB_2)R^{-1}B_2^T(\frac{\partial S}{\partial x_{n+1}})^T)\end{aligned} \tag{30}$$

in the interval $\tau\in[1,2]$.

Solving the equations (17-19) (when k=1) and (25-27) for $\tau\in[0,1)$ and the equations (20-22) (when k=2) and (28-30) for $\tau\in[1,2]$ together with the following boundary