MILP for FJSP

Indices and sets

i	jobs ($i, i' \in J$)
j	operations ($j, j' \in O$)
k	machines ($k \in M$)
J	the set of jobs
M	the set of machines
O	the set of operations
O_i	ordered set of operations of job i ($O_i \subseteq O$), where $O_{if_{(i)}}$ is the first and $O_{il_{(i)}}$ is the last element of O_i
M_j	the set of alternative machines on which operation j can be processed, ($M_j \subseteq M$)
$M_j \cap M_{j'}$	the set of machines on which operations j and j' can be processed

Parameters

t_{ijk}	the processing time of operation O_{ij} on machine k
L	a large number

Decision variables

X_{ijk}	1, if machine k is selected for operation O_{ij}; 0, otherwise
S_{ijk}	the starting time of operation O_{ij} on machine k
C_{ijk}	the completion time of operation O_{ij} on machine k
$Y_{iji'j'k}$	1, if operation O_{ij} precedes operation $O_{i'j'}$ on machine k; 0, otherwise
C_i	the completion time of job i
$C_{\max}$	maximum completion time over all jobs (makespan)

The proposed mathematical model is defined as follows:
Objective function: Minimize $C_{\max}$
Constraints:

$$\sum_{k \in M_j} X_{ijk} = 1 \quad \forall i \in J,\ \forall j \in O_i, \tag{1}$$

$$S_{ijk} + C_{ijk} \leqslant (X_{ijk}) \cdot L \quad \forall i \in J,\ \forall j \in O_i,\ \forall k \in M_j, \tag{2}$$

$$C_{ijk} \geqslant S_{ijk} + t_{ijk} - (1 - X_{ijk}) \cdot L \quad \forall i \in J,\ \forall j \in O_i,\ \forall k \in M_j, \tag{3}$$

$$S_{ijk} \geqslant C_{i'j'k} - (Y_{iji'j'k}) \cdot L \quad \forall i < i',\ \forall j \in O_i,\ \forall j' \in O_{i'},\ \forall k \in M_j \cap M_{j'}, \tag{4}$$

$$S_{i'j'k} \geqslant C_{ijk} - (1 - Y_{iji'j'k}) \cdot L \quad \forall i < i',\ \forall j \in O_i,\ \forall j' \in O_{i'},\ \forall k \in M_j \cap M_{j'}, \tag{5}$$

$$\sum_{k \in M_j} S_{ijk} \geqslant \sum_{k \in M_j} C_{i,j-1,k} \quad \forall i \in J,\ \forall j \in O_i - \{O_{if_{(i)}}\}, \tag{6}$$

$$C_i \geqslant \sum_{k \in M_j} C_{i,O_{il_{(i)}},k} \quad \forall i \in J, \tag{7}$$

$$C_{\max} \geqslant C_i \quad \forall i \in J, \tag{8}$$

and

$$X_{ijk} \in \{0,1\} \quad \forall i \in j,\ \forall j \in O_i,\ \forall k \in M_j,$$
$$S_{ijk} \geqslant 0 \quad \forall i \in j,\ \forall j \in O_i,\ \forall k \in M_j,$$
$$C_{ijk} \geqslant 0 \quad \forall i \in j,\ \forall j \in O_i,\ \forall k \in M_j,$$
$$Y_{iji'j'm} \in \{0,1\} \quad \forall i < i',\ \forall j \in O_i,\ \forall j' \in O_{i'},\ \forall k \in M_j \cap M_{j'},$$
$$C_i \geqslant 0 \quad \forall i \in j.$$

Constraints (1) make sure that operation O_{ij} is assigned to only one machine. If operation O_{ij} is not assigned to machine k, the constraints (2) set the starting and completion times of it on machine k equal to zero. Otherwise, the constraints (3) guarantee that the difference between the starting and the completion times is equal in the least to the processing time on machine k. Constraints (4) and (5) take care of the requirement that operation O_{ij} and operation $O_{i'j'}$ cannot be done at the same time on any machine in the set $M_j \cap M_{j'}$. Constraints (6) ensure that the precedence relationships between the operations of a job are not violated, i.e. the operation O_{ij} is not started before the operation $O_{i,j-1}$ has been completed. Constraints (7) determine the completion times (of the final operations) of the jobs, and constraints (8) determine the makespan.