4.1 − 1

Show that $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\lg n)$. Using substitution we want to prove that $T(n) \leqslant c\lg(n-b)$. Assume this holds for $\lceil n/2 \rceil$. We have:

$$\begin{aligned}
T(n) &\leqslant c\lg(\lceil n/2 - b \rceil) + 1 \\
&< c\lg(n/2 - b + 1) + 1 \\
&= c\lg(\frac{n - 2b + 2}{2}) + 1 \\
&= c\lg(n - 2b + 2) - c\lg 2 + 1 \\
&\leqslant c\lg(n - b)
\end{aligned}$$

The last inequality requires that $b \geqslant 2$ and $c \geqslant 1$.

4.2 − 1

Determine an upper bound on $T(n) = 3T(\lfloor n/2 \rfloor) + n$ using a recursion tree. We have that each node of depth i is bounded by $n/2^i$ and therefore the contribution of each level is at most $(3/2)^i n$. The last level of depth $\lg n$ contributes $\Theta(3^{\lg n}) = \Theta(n^{\lg 3})$. Summing up we obtain:

$$\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n \\
&\leqslant n + (3/2)n + (3/2)^2 n + \cdots + (3/2)^{\lg n - 1} n + \Theta(n^{\lg 3}) \\
&= n \sum_{i=0}^{\lg n - 1} (3/2)^i + \Theta(n^{\lg 3}) \\
&= n \cdot \frac{(3/2)^{\lg n} - 1}{(3/2) - 1} + \Theta(n^{\lg 3}) \\
&= 2(n(3/2)^{\lg n} - n) + \Theta(n^{\lg 3}) \\
&= 2n\frac{3^{\lg n}}{2^{\lg n}} - 2n + \Theta(n^{\lg 3}) \\
&= 2 \cdot 3^{\lg n} - 2n + \Theta(n^{\lg 3}) \\
&= 2n^{\lg 3} - 2n + \Theta(n^{\lg 3}) \\
&= \Theta(n^{\lg 3})
\end{aligned}$$

We can prove this by substitution by assumming that $T(\lfloor n/2 \rfloor) \leqslant c\lfloor n/2 \rfloor^{\lg 3} - c\lfloor n/2 \rfloor$. We obtain:

$$\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n \\
&\leqslant 3c\lfloor n/2 \rfloor^{\lg 3} - c\lfloor n/2 \rfloor + n \\
&\leqslant \frac{3cn^{\lg 3}}{2^{\lg 3}} - \frac{cn}{2} + n \\
&\leqslant cn^{\lg 3} - \frac{cn}{2} + n \\
&\leqslant cn^{\lg 3}
\end{aligned}$$

Where the last inequality holds for $c \geqslant 2$.